7) Mathematical Expectation: (E(x))

Definition:

[If x is a discrete random variable which can assume any of the values $x_1, x_2, \ldots x_n$ with respective probabilities p_i, $i = 1, 2 \ldots n$ then its mathematical expectation is defined as

discrete function:

$$E(x) = \sum_{i=1}^{n} p_i x_i, \quad \sum p_i = 1.] \qquad E(x) = \sum x P(x)$$

On the otherhand if x can assume any one of the values $x_1, x_2, \ldots x_n$ with respective probabilities $p_1, p_2 \ldots$ then

$$E(x) = \sum_{i=1}^{\infty} p_i x_i$$ provided the series is convergent.

If $\phi(x)$ be a function of the variate x, so that it takes the values $\phi(x_1), \phi(x_2) \ldots \phi(x_n)$ when x takes the values $x_1, x_2, \ldots x_n$ and if $p_1, p_2 \ldots p_n$ be the respective probabilities then the expected value of the function $\phi(x)$ is defined as,

$$E[\phi(x)] = p_1 \phi(x_1) + p_2 \phi(x_2) + \ldots p_n \phi(x_n)$$

$$= \sum_{i=1}^{n} p_i \phi(x_i), \quad \sum p_i = 1$$

If x is a continuous random variable then

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$E(x^r) = \mu_r$ (Raw moment)

$E(x) = \mu_1'$

$E(x)^r = \sum_i p_i x_i^r$ which is defined as μ_r' the r^{th} moment of the probability distribution.

Thus μ_r' (about origin) $= E(x^r)$.

In particular μ_1' (about origin) $= E(x)$

and μ_2' (about origin) $= E(x^2)$

Hence, Mean$(\bar{x}) = \mu_1'$ (about origin) $= E(x)$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$= E(x)^2 - [E(x)]^2 \rightarrow (a)$$

If $\phi(x) = (x - \bar{x})^r$, then,

$E(x - \bar{x})^r = \sum_i p_i (x_i - \bar{x})^r$ which is μ_r the r^{th}

moment about mean,

$$\mu_2 = E(x - \bar{x})^2$$

$$= \sum p_i (x_i - \bar{x})^2$$

$$= E[x - E(x)]^2 \text{ since } \bar{x} = E(x) \rightarrow (b)$$

(a) and (b) give the variance in terms of expectation.

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(1) Expectation of the sum of random variables:

Discrete random variable

(The mathematical expectations of sum of random variables is equal to the sum of their expectations.

symbolically if $x, y, z \ldots T$ are 'n' random variables then, $E(x+y+z+\ldots+T) = E(x) + E(y) + E(z) + \ldots + E(T)$)

Proof:-

Let x be a random variable. It assumes the values $x_1, x_2, \ldots x_n$ with respective probabilities $p_1, p_2, \ldots p_n$ and y be another random. It assumes the values $y_1, y_2, \ldots y_m$ with respective probabilities $p_1', p_2', \ldots p_m'$.

$$E(x) = \sum_{i=1}^{n} x_i P(x_i) \text{ and } E(y) = \sum_{j=1}^{m} y_j P(y_j) \rightarrow (1)$$

$$= \sum_{i=1}^{n} x_i p_i \qquad = \sum_{j=1}^{m} y_j p_j'$$

Let $P_{ij} = P(x = x_i, y = y_j)$. The sum $(x+y)$ is also a random variable which can take nm values $(x_i + y_j)$ $(i = 1, 2, \ldots n)$, $(j = 1, 2, \ldots m)$ Since any one of the values x_i $(i = 1, 2, \ldots n)$ can be associated with any one of the m values y_j, $j = 1, 2, \ldots m$.

$$E(x+y) = \sum_{i=1}^{n} \sum_{j=1}^{m} \cancel{P_{ij}} (x_i + y_j) . \cancel{P_i P_j'} P_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} P_{ij} x_i + \sum_{i=1}^{n} \sum_{j=1}^{m} P_{ij} y_j$$

$$= \sum_{i=1}^{n} x_i \sum_{j=1}^{m} P_{ij} + \sum_{j=1}^{m} y_j \sum_{i=1}^{n} P_{ij} \rightarrow (2)$$

Now x_i can occur in any of the mutually exclusive ways $(x_i + y_j)$ $j = 1, 2, \ldots m$. Hence by the addition theorem of probability we have,

$$P_i = P(x = x_i)$$

$$= P(x = x_i, y = y_1) + P(x = x_i, y = y_2) + \cdots + P(x = x_i, y = y_m).$$

$$= P_{i1} + P_{i2} + \cdots + P_{im} = \sum_{j=1}^{m} P_{ij} \rightarrow (3)$$

similarly, $P_j' = P[y = y_j] = \sum_{i=1}^{n} P_{ij}$

(ie) $P_j' = P[x = x_1, y = y_j] + P[x = x_2, y = y_j] + \cdots + P[x = x_n, y = y_j]$

$$= P_{1j} + P_{2j} + \cdots + P_{nj}$$

$$= \sum_{i=1}^{n} P_{ij} \rightarrow (4)$$

Putting (3) & (4) in (2) we get,

$$E(x+y) = \sum_{i=1}^{n} x_i p_i + \sum_{j=1}^{m} y_j P_j' = E(x) + E(y)$$

This result can be generalised to the case of more than two variables.

$$E(X+Y+Z) = E(X) + E(Y+Z)$$
$$= E(X) + E(Y) + E(Z)$$

Hence, we have,

$$E(X+Y+Z+\dots+T) = E(X) + E(Y) + E(Z) + \dots + E(T).$$

3) Expectation of the product of Independent random variables:

The mathematical expectations of a number of independent random variable is equal to the product of their expectations. symbolically if $X, Y, Z \dots T$ are n independent random variables. Then $E[XYZ \dots T] = E(X)E(Y)E(Z) \dots E(T)$.

Proof:

Let X be a random variable which assumes the values x_i $(i = 1, 2, \dots n)$ with respective probabilities $P_1, p_2, \dots p_n$ and Let Y be another independent random variables which assumes the values y_j $(j = 1, 2, \dots m)$ with respective probabilities $p_1', p_2' \dots p_m'$

Then $E(X) = \sum_{i=1}^{n} p_i x_i$ and $E(Y) = \sum_{j=1}^{m} p_j' y_j$

Let $p_{ij} = P[X = x_i, Y = y_j]$

Since X and Y are independent

$p_{ij} = P(X = x_i) \cdot P(Y = y_j)$

$= p_i p_j'$

The product XY is also a random variable which can assume mn values $x_i y_j$ $(i=1,2,\dots n)$ $(j=1,2,\dots m)$.

Then by definition,

$$E(XY) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j\, p_{ij}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j\, p_i p_j' = \sum_{i=1}^{n} p_i x_i \sum_{j=1}^{m} p_j' y_j$$

$$= E(X) \cdot E(Y)$$

For three independent random variable X, Y, Z

$$E(XYZ) = E(X) \cdot E(YZ) = E(X) \cdot E(Y) \cdot E(Z)$$

In general for 'n' independent random variable $X, Y, Z \dots T$, we have

$$E(XYZ \dots T) = E(X)\, E(Y)\, E(Z) \dots E(T).$$

If c is a constant then, $E(c) = c$

$$E(c) = \sum_{i=1}^{n} p_i \cdot c$$

$$= c \times \sum p_i = c \times 1 = c.$$

$\sum p_i = 1$ probability function

Theorem:

If x is a random variable and 'a' is a constant, then,

i) $E[a \cdot \phi(x)] = a E[\phi(x)]$

ii) $E[\phi(x) + a] = E[\phi(x)] + a$

where $\phi(x)$ is any function of x.

Proof:

If x can take the possible values $x_1, x_2 \dots x_n$ with respective probabilities $p_1, p_2 \dots p_n$ then by definition,

i) $E[a\,\phi(x)] = \sum_{i=1}^{n} a\phi(x_i)p_i$

$= a\sum_{i=1}^{n} \phi(x_i)p_i$

$= a \cdot E[\phi(x)]$

ii) $E[\phi(x)+a] = \sum_{i=1}^{n} [\phi(x_i)+a]p_i$

$= \sum_{i=1}^{n} [\phi(x_i)]p_i + \sum_{i=1}^{n} ap_i$

$= E[\phi(x)] + a \underline{\sum_{i=1}^{n} p_i}$ value (1)

$= E[\phi(x)] + a.$

---

(1): i) If $\phi(x) = x$ then $E(ax) = a.E(x)$

$E(x+a) = E(x)+a$

ii) If $\phi(x) = 1$, then $E(a) = a.$

Theorem:

If x is a random variable and 'a' and 'b' are constants then $E(ax+b) = aE(x)+b$.

Proof:

By definition we have,

$$E(ax+b) = \sum_{i=1}^{n}(ax_i+b)p_i = \sum_{i=1}^{n} ax_ip_i + \sum_{i=1}^{n} p_i b$$

$$= a\sum x_ip_i + b\sum_{i=1}^{n} p_i$$

$\sum p_i = 1$

$E(ax+b) = a.E(x)+b$

$V(x) = E[x-E(x)]^2$

If $a=1$, $b = -\bar{x} = -E(x)$, we get,

$E(x-\bar{x}) = E(x) - \bar{x} = \bar{x}-\bar{x} = 0$

Theorem:

If x is a random variable then $\underline{V(ax+b)} = a^2V(x)$ where a and b are constants.

let take $Y = ax+b$

Proof:

Let $Y = ax+b$

$$E(Y) = a.E(x)+b$$

$$= ax+b - [aE(x)+b]$$

$$Y-E(Y) = ax\cancel{+b} - a.E(x)\cancel{-b}$$

$$= a[x-E(x)]$$

Squaring and taking expectations of both sides, we get, $[Y-E(Y)]^2 = a^2[x-E(x)]^2$

$$E[Y-E(Y)]^2 = a^2 E[X-E(x)]^2$$

$$V(y) = a^2 V(x)$$

$$V(ax+b) = a^2 V(x)$$

$V(a) = 0$

$V(ax) = a^2 V(x)$

$V(x) = E[x-E(x)]^2$

$V(Y) = E[Y-E(Y)]^2$

Cor:

If $b=0$, $V(ax) = a^2 V(x)$

If $a=0$, $V(b) = 0$

If $a=1$, $V(x+b) = V(x)$

Expectation of a linear combination of random variables:

Let $x_1, x_2 \dots x_n$ be any 'n' random variable and if $a_1, a_2 \dots a_n$ are any 'n' constants then,

$$E\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i E(x_i)$$

Proof:

$$E\left(\sum_{i=1}^{n} a_i x_i\right) = E[a_1x_1 + a_2x_2 + \dots a_nx_n]$$

$$= E[a_1x_1] + E[a_2x_2] + \dots + E[a_nx_n]$$

$$= a_1E(x_1) + a_2E(x_2) + \dots + [a_nE(x_n)]$$

$$= \sum_{i=1}^{n} a_i E(x_i)$$

Expectation of a continuous R.V:

If x is a continuous random variable with p.d.f $f(x)$ then,

$$F(x) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{\infty} x \cdot dF(x)$$

Provided the integral exists.

Note:

If g is a function such that $g(x)$ is a random variable and $E[g(x)]$ exists, then,

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx.$$

Mathematical expectation.

Properties of expected value: (continuous variable) 1) State the properties. 2) Proof.

i) If c is a constant $E(C) = c$.

Proof:

$\because E(x) = \int_{-\infty}^{\infty} f(x) dx = 1$

$$E(C) = \int_{-\infty}^{\infty} c f(x)\, dx = c \underbrace{\int_{-\infty}^{\infty} f(x)\, dx}_{1}$$

$$= c \times 1$$

$$E(C) = c.$$

ii) If a is a constant and x is a random variable $\phi(x)$ is any function of x, $E[a\phi(x)] = aE[\phi(x)]$.

Proof:

$$E[a\phi(x)] = \int_{-\infty}^{\infty} a\phi(x) f(x) dx$$

$$= a\int_{-\infty}^{\infty} \phi(x) f(x) dx = aE[\phi(x)]$$

*In particular $E.[ax] = a.E(x)$.

iii) If x is a r.v and $g_1(x)$ then $g_2(x)$ are two functions of x then $E[g_1(x) + g_2(x)] = E[g_1(x)] + E[g_2(x)]$

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assuming that the expectations of $g_1(x)$ and $g_2(x)$ exists.

Proof:

$$E[g_1(x)+g_2(x)] = \int_{-\infty}^{\infty} [g_1(x) + g_2(x)] f(x)\, dx$$

$$= \int_{-\infty}^{\infty} g_1(x) f(x)\, dx + \int_{-\infty}^{\infty} g_2(x) f(x)\, dx$$

$$= E[g_1(x)] + E[g_2(x)].$$

iv) $E[g_1(x)] \leq E[g_2(x)]$ if $g_1(x) \leq g_2(x)$.

$\therefore g_2$ is a greater value.

Proof:

$$E[g_2(x)] - E[g_1(x)] = E[g_2(x) - g_1(x)] \text{ for all}$$

$$= \int_{-\infty}^{\infty} [g_2(x) - g_1(x)] f(x)\, dx$$

$$\geq 0 \quad \text{since } g_1(x) \leq g_2(x)$$

$$\therefore E[g_2(x)] \geq E[g_1(x)]$$

(ie) $E[g_1(x)] \leq E[g_2(x)]$.

v) $|E[g(x)]| \leq E[|g(x)|]$.

Proof:

$(= \neq)$ changed

$$|E[g(x)]| = \left| \int_{-\infty}^{\infty} g(x) f(x)\, dx \right|$$

$$\leq \int_{-\infty}^{\infty} |g(x)| f(x)\, dx.$$

$$\leq E[|g(x)|]$$

$$\therefore |E[g(x)]| \leq E[|g(x)|].$$

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CONDITIONAL EXPECTATION:-

The conditional expectation of a discrete random variable X given Y=y is

$$E[X/Y=y] = \sum_{i=1}^{\infty} x_i P[X=x_i / Y=y_i]$$

The conditional expectation of a discrete random variable Y given X=x is

$$E[Y/X=x] = \sum_{i=1}^{\infty} y_i P[Y=y_i / X=x_i]$$

The conditional expectation of a continuous random variable X given Y=y is

$$E[X/Y=y] = \frac{\int_{-\infty}^{\infty} x f(x,y)\,dx}{f(y)}$$

The conditional expectation of a continuous random variable Y given X=x is

$$E[Y/X=x] = \frac{\int_{-\infty}^{\infty} y f(x,y)\,dy}{f(x)}.$$

Prove that:-

$E[E(X/Y)] = E(X)$ & $E[E(Y/X)] = E(Y)$.

Proof:-

By using conditional expectation of a continuous random variable.

$$E(X/Y) = \frac{\int_{-\infty}^{\infty} x f(x,y)\,dx}{}$$

$$E(X/Y) = \frac{\int_{-\infty}^{\infty} x\, f(x,y)\, dx}{f(y)} = Z(y)$$

$$E[Z(y)] = \int Z(y) f(y)\, dy.$$

$$E[Z(y)] = \int \frac{\int x f(x,y) dx}{f(y)} \cdot f(y) dx.$$

$$= \iint x\, f(x,y)\, dx\, dy$$

$$= \int x \left[\int f(x,y)\, dy\right] dx$$

$$= \int x\, f(x)\, dx$$

$$E[E(X/Y)] = E(X)$$

Hence proved $\rightarrow (1)$

$$E[E(Y/X)] = E(Y)$$

By using conditional expectation of a continuous random variable

$$E(Y/X) = \frac{\int_{-\infty}^{\infty} y\, f(x,y)\, dy}{f(x)}$$

$$E(Y/X) = \frac{\int_{-\infty}^{\infty} y\, f(x,y)\, dy}{f(x)}$$

$$E(Y/X) = \frac{\int_{-\infty}^{\infty} y\, f(x,y)\, dy}{f(x)} = Z(x)$$

$$E[Z(x)] = \int Z(x) f(x)\, dx.$$

$$= \int \frac{\int y\, f(x,y) dy}{f(x)} \cdot f(x)\, dx$$

$$= \iint y\, f(x,y)\, dy\, dx$$

$$= \int y \left[\int f(x,y)\, dx \right] dy.$$

$$E[Z(x)] = \int y\, f(y)\, dy.$$

$$E[E(Y/x)] = E(Y).$$

Hence proved $\rightarrow$ (2).

Note:- 1

1. If X & Y are independent then $[E(X/Y)] = E(X)$.

proof:-

$$E(X/Y) = \frac{\int x\, f(x,y)\, dx}{f(y)}$$

$$= \int x\, f(x) \cdot f(y)\, dx \,/ f(y)$$

$$= \int x\, f(x)\, dx.$$

$$E(X/Y) = E(X)$$

Hence proved.

Note:- 2

2. If X & Y are independent then $E[X - E(X/Y)] = 0$.

Proof:-

Since X & Y are independent we can write $E[X/Y] = E(X)$

$$E[X - E(X/Y)] = E[X - E(X)]$$

$$= E(X) - E[E(X)]$$

$$= E(X) - E(X)$$

$$E[X - E(X/Y)] = 0.$$

Hence proved.

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BERNOULLI DISTRIBUTION:

A random variable x which takes the values 1 and 0 with probability p and q respectively. i.e $P(x=1)=p$ and $P(x=0)=q$. $q=1-p$ is called a Bernoulli variate and is said to have a Bernoulli distribution.

BERNOULLIAN TRIALS:

When trials are repeated in such a way that the probability of success in each trial is constant and is independent of previous happening. They are called Bernoullian trials.

Remark:

Sometimes the two values are $+1, -1$ instead of 1 and 0.

MOMENTS OF BERNOULLI DISTRIBUTION:

The r^{th} moment about origin

$$\mu'_r = E(x^r) = 0^r \cdot q + 1^r \cdot p = p$$

$$\mu'_1 = E(x) = p$$

$$\mu'_2 = E(x^2) = p$$

$$\mu_2 = var(x) = \mu'_2 - (\mu'_1)^2$$

$$= p - p^2 = p(1-p)$$

$$= pq.$$

$p^x \; q^{n-x}$

BINOMIAL DISTRIBUTION:

Consider a series of 'n' independent Bernoullian trials in which the probability 'p' of success in any trial is constant for each trial. Then $q = 1 - p$ is the probability of failure in any trial the probability of 'x' success and consequently $(n-x)$ failures in 'n' independent trials is given by the compound probability theorem.

If we consider the series SSFSFFFS... FSF..., where F denotes failures and S denote successes.

By compound probability theorem,

$P(SSFSFFFS \ldots FSF)$

$= P(S).P(S).P(F)P(S)P(F)P(F)P(F)P(S) \ldots$

$\ldots P(F)\,P(S)\,P(F)$

$= p.p.q.p.q.q.q.p \ldots q.p.q.$

$= \underbrace{p.p.p \ldots p}_{x \text{ factors}} \quad \underbrace{q.q.q \ldots q}_{(n-x) \text{ factors}}$

$= p^x q^{n-x}$

But x successes in 'n' trials can occur in nC_x ways and the probability for each of these ways is $p^x . q^{n-x}$. Hence the probability of 'x' successes in 'n' trials is given by the addition theorem of

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probability.

$$nC_x \cdot p^x \cdot q^{n-x}.$$

The probability distribution of the number of successes so obtained is called Binomial probability distribution, because the probabilities of 0, 1, 2 ... n successes are,

$$\underset{nC_0p^0q^{n-0}}{q^n},\ \underset{x=1}{(n)}\,q^{n-1}p,\ \underset{x=2}{(n)}\,q^{n-2}p^2, \dots \underset{x=n}{p^n}$$ are

the successive terms of the Binomial expansion $(q+p)^n$.

DEFINITION:

A Random variable X is said to follow binomial distribution if it assumes only non-negative values and its probability mass function is given by,

$$P(X=x) = b(x;n,p) = nC_x\, p^x q^{n-x};$$
$$x = 0, 1, 2 \dots n$$

The two independent constants n and p are known as the parameters of the distribution.

Binomial distribution is a discrete distribution as x can take only the integral values 0, 1, 2 ... n Any variable which follows binomial distribution is known as binomial variate.

Remarks:-

i) The assignment of probabilities is permissible because

$$\sum P(X=x) = \sum_{x=0}^{n} P(x)$$

$$= \sum_{x=0}^{n} nC_x \, p^x q^{n-x}$$

$$= q^n + (n) p q^{n-1} + (n) p^2 q^{n-2} + \dots + p^n$$

$$= (q+p)^n = 1.$$

ii) If the experiment is repeated N times the frequency function of the binomial distribution is given by,

$$f(x) = N.P(x)$$

$$= N.nC_x \, p^x q^{n-x}$$

$$x = 0, 1, 2 \dots n$$

and the expected frequencies of 0, 1, 2, ... n successes are the successive terms of the binomial expansion $N(q+p)^n$.

MOMENTS:-

The first four moments about origin of Binomial distribution are obtained as follows.

$$\mu_1' = Mean = E(X) = \sum x.P(x)$$

$$= \sum_{x=0}^{n} x . nC_x . p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \cancel{x} \frac{n(n-1)!}{\cancel{x}(x-1)!\,(n-x)!} p \cdot p^{x-1} q^{n-x}$$

$$= np \sum_{x=1}^{n} {}^{n-1}C_{x-1}\, p^{x-1} q^{n-x}$$

$$= np(q+p)^{n-1}$$

$$= np \qquad (q+p)=1$$

$$\mu_2' = E(X^2) = \sum_{x=0}^{n} x^2\, nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \{x(x-1)+x\}\, nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)\, nC_x\, p^x q^{n-x} + \sum_{x=0}^{n} x\, nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{x!(n-x)!} p^x q^{n-x} + E(X)$$

$$= \sum_{x=0}^{n} \frac{x(x-1)n(n-1)(n-2)!}{x(x-1)(x-2)!(n-x)!} \cdot p^2 p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 (q+p)^{n-2} + np$$

$$= \boxed{n(n-1)p^2 + np}$$

$$\mu_3' = E(X^3) = \sum_{x=0}^{n} x^3 P(x)$$

$$= \sum_{x=0}^{n} \{x(x-1)(x-2) + 3x(x-1) + x\}\, nC_x\, p^x q^{n-x}$$

$$P(x)$$

$$= \sum_{x=0}^{n} x(x-1)(x-2) \underbrace{nC_x\, p^x q^{n-x}}_{P(x)} + 3 \cdot \sum_{x=0}^{n} x(x-1) \underbrace{nC_x\, p^x q^{n-x}}_{P(x)} + \sum x\, \underline{nC_x\, p^x q^{n-x}}$$

$$= \sum_{x=0}^{n} x(x-1)(x-2) \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$+3n(n-1)p^2 + np$$

$$= \sum_{x=0}^{n} \frac{x(x-1)(x-2)\, n(n-1)(n-2)(n-3)!}{x(x-1)(x-2)(x-3)!\,(n-x)!} p^3$$

$$+3n(n-1)p^2 + np \qquad (q+p)^{n-3}$$

$$= n(n-1)(n-2) \boxed{p^3 \sum_{x=3}^{n} \binom{n-3}{x-3} p^{x-3} q^{n-x}}$$

$$+3n(n-1)p^2 + np..$$

$$= n(n-1)(n-2)p^3(q+p)^{n-3} + 3n(n-1)p^2 + np$$

$$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$

$$\mu_4' = \sum_{x=0}^{n} x^4 . p^x = E(x^4)$$

$$= \sum_{x=0}^{n} \{ x(x-1)(x-2)(x-3) + 6x(x-1)(x-2)$$

$$+7x(x-1) + x \} {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)(x-2)(x-3)\, {}^nC_x p^x q^{n-x}$$

$$+ 6 \sum_{x=0}^{n} x(x-1)(x-2)\, {}^nC_x p^x q^{n-x}$$

$$+ 7 \sum_{x=0}^{n} x \cdot {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)(x-2)(x-3) \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$+ 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$

$$= \sum_{x=0}^{n} \frac{x(x-1)(x-2)(x-3)\, n(n-1)(n-2)(n-3)(n-4)!}{x(x-1)(x-2)(x-3)(x-4)!}$$

$$p^4 p^{x-4} q^{n-x} + 6n(n-1)(n-2)p^3$$

$$+ 7n(n-1)p^2 + np.$$

$= n(n-1)(n-2)(n-3)p^4 \sum_{x=4}^{n} \binom{n-4}{x-4} p^{x-4} q^{n-x}$

$+6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$

$= n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3$

$+7n(n-1)p^2 + np$

Hence Mean $= np$.

[variance $\Rightarrow \mu_2 = \mu_2' - (\mu_1')^2$

$= n(n-1)p^2 + np - (np)^2$

$= (n^2-n)p^2 + np - n^2p^2$

$= n^2p^2 - np^2 + np - n^2p^2$

$= np - np^2$

$= np(1-p)$

$= npq$.]

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$

$-3\{n(n-1)p^2 + np\}\{np\} + 2(np)^3$

$= (n^2-n)(np^3 - 2p^3) + 3n^2p^2 - 3np^2 + np$

$-3\{n^2p^2 - np^2 + np\}\{np\} + 2n^3p^3.$

$= 2np^3 - 3np^2 + np$

$= np\{2p^2 - 3p + 1\}$

$= np\{2p^2 - 2p - p + p + q\}$

$= np\{2p^2 - 2p + q\}$ $\qquad q+p=1$

$= np\{2p(p-1) + q\}$ $\qquad p-1=q$

$= np\{2p(-q) + q\}$

$$= npq\{-2p+\}$$

$$= npq\{-2p+p+q\}$$

$$\mu_3 = npq(q-p)$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$= \{n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np\} + 4\{n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np\}\{np\} + 6\{n(n-1)p^2 + np\}\{n^2p^2\} - 3n^4p^4.$$

$$= np[\{(n-1)(n-2)(n-3)p^3 + 6(n-1)(n-2)p^2 + 7(n-1)p + 1\} - 4\{n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np\} + 6\{n(n-1)p^2 + np\}\{np\} - 3n^3p^3].$$

$$= np[(n^2-3n+2)(np^3-3p^3) + 6n^2p^2 - 18np^2 + 12p^2 + 7np - 7p + 1 - 4n^3p^3 + 12n^2p^3 - 8np^3 - 12n^2p^2 + 12np^2 + 4np + 6n^3p^3 - 6n^2p^3 + 6n^2p^2 - 3n^3p^3]$$

$$= np[n^3p^3 - 3n^2p^3 - 3n^2p^3 + 9np^3 + 2np^3 - 6p^3 + 6n^2p^2 - 18np^2 + 12p^2 + 7np - 7p + 1 - 4n^3p^3 + 12n^2p^3 - 8np^3 - 12n^2p^2 + 12np^2 + 4np + 6n^3p^3 - 6n^2p^3 + 6n^2p^2 - 3n^3p^3]$$

$= np[3np^3 - 6p^3 - 6np^2 + 12p^2 + 3np - 7p + 1]$

$= np[3np\{p^2 - 2p + 1\} - 6p^3 + 12p^2 - 6p - p$
$+ p + q]$

$= np[3np(p-1)^2 - 6p(p^2 - 2p + 1) + q]$

$= np[3np(-q)^2 - 6p(p-1)^2 + q]$

$= np[3npq^2 - 6pq^2 + q]$

$= npq[3npq - 6pq + 1]$

$= npq[3pq(n-2) + 1]$.

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\{npq(q-p)\}^2}{(npq)^3}$$

$$= \frac{(npq)^2(q-p)^2}{(npq)^3}$$

$$= \frac{(q-p)^2}{npq} = \left(\frac{(1-p-p)^2}{npq}\right)$$

$$\beta_1 = \frac{(1-2p)^2}{npq}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{npq[3pq(n-2)+1]}{(npq)^2}$$

$$= \frac{3pq(n-2)+1}{npq}$$

$$= \frac{3npq - 6pq + 1}{npq}$$

$$= 3 + \frac{1 - 6pq}{npq}$$

$$\gamma_1 = \sqrt{\beta_1}$$

$$= \frac{\sqrt{(1-2p)^2}}{\sqrt{npq}}$$

$$= \frac{1-2p}{\sqrt{ppq}}$$

$$\gamma_2 = \beta_2 - 3$$

$$= 3 + \frac{1-6pq}{npq} - 3$$

$$= \frac{1-6pq}{npq}$$

Hence mean $= np$, variance $= npq$

$$\beta_1 = (1-2p)^2/npq$$

$$\beta_2 = 3 + \frac{1-6pq}{npq}$$

$$= (1-2p)/\sqrt{npq}$$

$$= (1-6pq)/npq.$$

$$\mu_3 = npq(q-p)$$

$$\mu_4 = npq[1+3pq(n-2)]$$

RECURRENCE FORMULA FOR MOMENTS:-

All the central moments of a Binomial distribution can be determined from the recurrence formula

$$\mu_{r+1} = pq\left[nr \cdot \mu_{r-1} + \frac{d\mu_r}{dp}\right]$$

Proof:-

M^{rth} - Moment about Mean is μ_r.

$$\mu_r = E[x-\mu]^r = E[x-np]^r$$

$$= \sum_{x=0}^{n} (x-np)^r P(X=x)$$

$$= \sum_{x=0}^{n} (x-np)^r \underbrace{nC_x p^x q^{n-x}}_{uv}$$

Diff w.r to p,

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} nC_x [\{r(x-np)^{r-1}(-n) p^x q^{n-x}\}$$
$$+ (x-np)^r \{x p^{x-1} q^{n-x} + p^x (n-x) q^{(n-x-1)} (-1)$$

$$= -nr \sum_{x=0}^{n} nC_x (x-np)^{r-1} p^x q^{n-x}.$$
$$+ \sum nC_x [(x-np)^r p^x q^{n-x} [x p^{-1} - (n-x) q^{-1}]$$

$$= -nr\mu_{r-1} + \sum nC_x (x-np)^r p^x q^{n-x}$$
$$\left\{ \frac{x}{p} - \frac{(n-x)}{q} \right\}$$

$$= -nr\mu_{r-1} + \sum nC_x (x-np)^r p^x q^{n-x}$$
$$\left\{ \frac{xq - np + xp}{pq} \right\}$$

$$= -nr\mu_{r-1} + \sum nC_x (x-np)^r p^x q^{n-x}$$
$$\frac{1}{pq}\{x(p+q) - np\}$$

$$= -nr\mu_{r-1} + \frac{1}{pq} \sum (x-np)^{r+1} nC_x p^x q^{n-x}$$

$$= -nr\mu_{r-1} + \frac{1}{pq} \mu_{r+1}$$

$$\Rightarrow \frac{1}{pq} \mu_{r+1} = nr\mu_{r-1} + \frac{d}{dp} \mu_r$$

$$\mu_{r+1} = pq\left[nr\mu_{r-1} + \frac{d}{dp}\mu_r\right]$$

when $r=1$,

$$\mu_2 = pq\left[n\mu_0 + \frac{d}{dp}\mu_1\right] \qquad \because \mu_0 = 1$$

$$\mu_1 = 0$$

$$= pq[n+0] = npq.$$

$r=2$,

$$\mu_3 = pq\left[2n\mu_1 + \frac{d}{dp}\mu_2\right]$$

$$= pq\left[0 + \frac{d}{dp}npq\right]$$

$$= npq\left[\frac{d}{dp}(pq)\right]$$

$$= npq\left[\frac{d}{dp}\{p(1-p)\}\right]$$

$$= npq\left[\frac{d}{dp}(p-p^2)\right]$$

$$= npq(1-2p)$$

$$= npq(p+q-2p)$$

$$= npq(q-p).$$

$$\mu_4 = pq\left[3n\mu_2 + \frac{d\mu_3}{dp}\right]$$

$$= pq\left[3n(npq) + \frac{d}{dp}(npq(q-p))\right]$$

$$= pq\left[3n^2pq + n.\frac{d}{dp}\{pq(q-p)\}\right]$$

$$= npq\left[3npq + \frac{d}{dp}(pq^2 - p^2q)\right]$$

$$= npq\left[3npq + \frac{d}{dp}\{p(1-p)^2 - p^2(1-p)\}\right]$$

$$= npq\left[3npq + \frac{d}{dp}\{2p^3 - 3p^2 + p\}\right]$$

$$= npq[3npq + 6p^2 - 6p + 1]$$

$$= npq\,[3npq + 1 - 6p(1-p)]$$

$$= npq\,[3npq + 1 - 6pq]$$

$$= npq[3pq(n-2)+1].$$

Note:-

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\{npq(q-p)\}^2}{(npq)^3}$$

$$= \frac{(q-p)^2}{npq}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{npq[3pq(n-2)+1]}{(npq)^2}$$

$$= \frac{3npq - 6pq + 1}{(npq)}$$

$$= 3 + \frac{1-6pq}{npq}$$

When p and q are finite and n becomes large $\beta_1 \to 0$ and $\beta_2 \to 3$ when p becomes small and 'n' becomes large, in that case $np \to$ a finite quantity m,

$$\beta_1 \to \frac{1}{m} \text{ and } \beta_2 \to 3 + \frac{1}{m}.$$

Mode of the Binomial distribution

The value of the variate which corresponds to the maximum frequency (or probability) is called the mode or modal value.

The probability of x success is given by

$$P(x) = nC_x p^x q^{n-x}.$$

$$P(x-1) = nC_{x-1} p^{x-1} q^{n-x+1}.$$

$$\frac{P(x)}{P(x-1)} = \frac{nC_x p^x q^{n-x}}{nC_{x-1} p^{x-1} q^{n-x+1}}$$

$$= \frac{n!(x-1)!(n-x+1)!\,p^x q^{n-x}}{x!(n-x)!\,n!\,p^{x-1} q^{n-x+1}}$$

$$= \frac{(x-1)!(n-x+1)!\,p^x q^{n-x}}{x(x-1)!(n-x)!\,p^x q^{n-x} p^{-1} q}$$

$$= \frac{(n-x+1)}{x}\cdot\frac{p}{q} = \frac{(x-x+1)p}{xq} = \frac{(n-x+1)p}{xq}$$

$$= \frac{xq + (n-x+1)p - xq}{xq}$$

$$= 1 + \frac{(n-x+1)p - xq}{xq}$$

$$= 1 + \frac{np - xp + p - xq}{xq}$$

$$= 1 + \frac{(n+1)p - x(p+q)}{xq}$$

$$\frac{P(x)}{P(x-1)} = 1 + \frac{(n+1)p - x}{xq}$$

We discuss the following two cases.

Case I :-

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1 Moment Generating function of Binomial distribution:-

$$M_X(t) = E[e^{tx}] = \sum_{x=0}^{n} e^{tx}\, nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} e^{tx} \cdot nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} (p \cdot e^t)^x\, nC_x\, q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x\, (p \cdot e^t)^x\, q^{n-x}$$

$x = 0, 1 \ldots n.$

$$= nC_0 (p \cdot e^t)^0 q^n + nC_1 (p \cdot e^t)^1 q^{n-1}$$

$$\ldots + nC_n (p \cdot e^t)^n q^0$$

$$= (q + pe^t)^n.$$

The moments can be obtained from the moment generating function as follows:

$$M_X(t) = (q + p \cdot e^t)^n.$$

$$\mu_r' = \left[\frac{d^r}{dt} M_x(t)\right]_{t=0}$$

$r = 1$, $\mu_1' = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$

$$= \left(\frac{d}{dt}\left[(q + pe^t)\right]^n\right)$$

$$= \left[n (q + pe^t)^{n-1} p \cdot e^t\right]_{t=0}.$$

$$= \left[np(q + pe^t)^{n-1} e^0\right]_{t=0}$$

$$\mu_1' = np.$$

$r=2,\quad \mu_2' = \left[\frac{d\, M_x(t)}{dt^2}\right]_{t=0}$

$$= \frac{d}{dt}\left[\underbrace{np(q+p.e^t)^{n-1}}_{u} \cdot \underbrace{e^t}_{v}\right]_{t=0}$$

$$= \left[np(q+p.e^t)^{n-1} + n(n-1)p\,(q+p.e^t)^{n-2} e^t\, p.e^t\right]_{t=0}$$

$$\mu_2' = np + n(n-1)p^2.$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$= np - n(n-1)p^2 - (np)^2$$

$$= np + n^2p^2 - np^2 - n^2p^2$$

$$= np(1-p) = npq.$$

$$\mu_3' = \frac{d}{dt}\left[\frac{d^2}{dt^2}(M_x)(t)\right]$$

$$= \frac{d}{dt}\left[np(q+pe^t)^{n-1} e^t + n(n-1)p^2 (q+pe^t)^{n-2} e^t e^t\right]$$

$$= \Big[np(q+p.e^t)^{n-1}e^t + n(n-1)p^2e^{2t}(q+p.e^t)^{n-2}$$
$$+ 2n(n-1)p^2e^{2t}(q+p.e^t)^{n-2}$$
$$+ n(n-1)(n-2)p^2e^{2t}(q+p.e^t)^{n-3}pe^t\Big]_{t=0}$$

$$= np + n(n-1)p^2 + 2n(n-1)p^2 + n(n-1)(n-2)p^3$$

$$= np + 3n(n-1)p^2 + n(n-1)(n-2)p^3$$

$$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$

$$\mu_4' = \left[\frac{d}{dt}\mu_3'\right]_{t=0}$$

$$= [np(q+p.e^t)^{n-1} e^t + n(n-1)p^2 e^{2t} (q+p.e^t)^{n-2} + 2n(n-1)p^2 e^{2t} (q+pe^t)^{n-2} + n(n-1)(n-2)p^2 e^{2t} (q+pe^t)^{n-2}$$

$$+ p.e^t + 2n(n-1)p^2 .2.e^{2t} (q+p.e^t)^{n-2}$$

$$+ 2n(n-1)(n-2)p^2 e^{2t} (q+p.e^t)^{n-3} p.e^t$$

$$+ n(n-1)(n-2)p^3 .3.e^{3t} (q+p.e^t)^{n-3}$$

$$+ n(n-1)(n-2)p^3. e^{3t} (n-3)(q+p.e^t)^{n-4} p.e^t]_{t=0}$$

$$= np + n(n-1)p^2 + 2n(n-1)p^2 + n(n-1)(n-2)p^3 + 4n(n-1)p^2 + 2n(n-1)(n-2)p^3 + 3n(n-1)(n-2)p^3 + n(n-1)(n-2)(n-3)p^4.$$

$$\mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np.$$

$\therefore$ $\mu_1' = np$

$\mu_2' = n(n-1)p^2 + np$

$\mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$

$\mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np.$

From these raw moments we get the central moments.

Always $\mu_1 = 0$

$\mu_2 = npq$

$\mu_3 = npq(q-p)$

$\mu_4 = npq[3pq(n-2)+1]$

Additive property of Binomial Random variable:-

If x_1 and x_2 are two independent Binomial random variables with parameters (p, n_1) and (p, n_2) then $x_1 + x_2$ is a binomial random variable with parameters (p, n_1+n_2)

$$M_{x_1}(t) = (q + p.e^t)^{n_1}$$

$$M_{x_2}(t) = (q+p.e^t)^{n_2}$$

Since x_1 and x_2 are independent random variables.

$$\therefore M_{x_1+x_2}(t) = M_{x_1}(t) \cdot M_{x_2}(t)$$

$$= (q+p.e^t)^{n_1} (q+p.e^t)^{n_2}$$

$$= (q+p.e^t)^{n_1+n_2}$$

which is the moment generating function of the binomial variable with parameter p and $n = n_1+n_2$.

Hence by the uniqueness theorem of moment generating function, x_1+x_2 is also a Binomial variate with parameters (p, n_1+n_2)

$$\therefore x_1 + x_2 \sim B(p, n_1+n_2).$$

Assume that on the average one telephone number out of fifteen called between 2 p.m and 3 p.m on week days is busy. What is the probability that if six randomly selected telephone numbers are called?

i) not more than three will be busy.

ii) atleast 3 of them will be busy.

P = probability that a telephone number is busy between 2 p.m and 3 p.m $= \frac{1}{15}$

The probability that out of six randomly selected telephone numbers x are busy is

$p = 1/15$

$q = 1-p$

$q = 1 - 1/15$

$= \frac{14}{15}$

$$P(X=x) = (6Cx)\, p^x (q)^{6-x}$$

$$= 6Cx \left(\frac{1}{15}\right)^x \left(\frac{14}{15}\right)^{6-x}$$

i) Probability that not more than three numbers are busy is

$$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= 6C_0 \left(\frac{1}{15}\right)^0 \left(\frac{14}{15}\right)^6 + 6C_1 \left(\frac{1}{15}\right)^1 \left(\frac{14}{15}\right)^5 + 6C_2 \left(\frac{1}{15}\right)^2 \left(\frac{14}{15}\right)^4 + 6C_3 \left(\frac{1}{15}\right)^3 \left(\frac{14}{15}\right)^3$$

$$= 6C_0 \left(\frac{14^6}{15^6}\right) + 6C_1 \left(\frac{14^5}{15^6}\right) + 6C_2 \left(\frac{14^4}{15^6}\right) + 6C_3 \left(\frac{14^3}{15^6}\right)$$

$$= \frac{1}{15^6}\left[1 \cdot 14^6 + 6(14)^5 + \left(\frac{6\times5}{1\times2}\right)(14)^4 + \left(\frac{6\times5\times4}{1\times2\times3}\right)(14)^3\right].$$

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$$= \frac{14^6}{15^6}\left[14^3 + 6(14)^2 + 15(14) + 20\right]$$

$$= \frac{2744}{11390625}\left[2744 + 1176 + 210 + 20\right]$$

$$= \frac{2744(4150)}{11390625} = \frac{11387600}{11390625}$$

$$= 0.9997.$$

ii) probability that atleast 3 numbers are busy is

$$P(X \geq 3) = P(X=3) + P(X=4) + P(X=5) + P(X=6)$$

$$= \frac{1}{15^6}\left[6C_3(14)^3 + 6C_4(14)^2 + 6C_5(14) + 1\right]$$

$$= \frac{1}{15^6}\left[20(14^3) + (15 \times 14^2) + (6 \times 14) + 1\right]$$

$$= \frac{1}{11390625}\left[54880 + 2940 + 84 + 1\right]$$

$$= \frac{57905}{11390625}$$

$$= 0.0051.$$

POISSON DISTRIBUTION

Poisson distribution is a limiting case of the binomial distribution under the following conditions.

i) n, the number of trials is indefinitely large ie, $n \to \infty$. (ii) (back side)

ii) neither p nor q is small, ie $p \to 0$

iii) $np = \lambda$ (say) is finite.

Thus $p = \lambda/n$, $q = 1 - \lambda/n$ where λ is a positive real number.

The probability of x successes in a series of n independent trials is

$$P(X=x) = b(x; n, p) \qquad X = 0, 1, 2, \ldots n$$

$$= nC_x \, p^x q^{n-x}$$

$$= nC_x \, p^x (1-p)^{n-x}$$

$$= nC_x \left(\frac{p}{1-p}\right)^x (1-p)^n.$$

$np = \lambda$, $p = \lambda/n$.

$$P(X=x) = \frac{n(n-1)(n-2)\ldots(n-(x+1))\cancel{(n-x)!}}{x!\,\cancel{(n-x)!}}$$

$$\cdot \frac{(\lambda/n)^x (1-p)^n}{(1-\lambda/n)^x} \qquad \frac{n^x \cdot \lambda^x (1-x)^n}{n^x x! (1-\lambda/n)^x}$$

$$= \frac{n^x(1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{x-1}{n})\lambda^x (1-p)^n}{n^x \, x! \,(1-\lambda/n)^x}$$

Now $(1-p)^n = 1 - np + \frac{n(n-1)}{2!}p^2 - \frac{n(n-1)(n-2)}{3!}p^3 + \ldots$

$np = \lambda$.

$$= 1 - \lambda + \frac{n^2\left(1-\frac{1}{n}\right)p^2}{2!} - \frac{n^2\left(1-\frac{1}{n}\right)n\left(1-\frac{2}{n}\right)}{3!}p^3 + \cdots$$

$$= 1 - \lambda + \frac{\lambda^2}{2!}\left(1-\frac{1}{n}\right) - \frac{\lambda^3}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots$$

$$\lim_{n\to\infty}(1-p)^n = 1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \cdots$$

$$\lim_{n\to\infty} b(x, n, p) = e^{-\lambda}$$

$$\lim_{n\to\infty} p(x=x) = \lim_{n\to\infty} \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right) \cdot \lambda^x e^{-\lambda}}{x!}$$

$$P(X=x) = \frac{1}{x!}\lambda^x e^{-\lambda}, \quad p(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$P(X=x) = P(x, \lambda) \text{ where } \lambda = np \text{ as } n\to\infty$$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x} = 1$$

$$\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^{-x+1/2} = 1$$

$$\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n = e^{-x}$$

Definition:-

A random variable X is said to follow a poisson distribution if it assumes only non-negative values and its probability mass function is given by

$$P(x, \lambda) = P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, \ldots$$
$$= 0, \text{ otherwise}$$

Here λ is known as the parameter of the distribution and $\lambda > 0$.

Poisson distribution occurs when there are events which do not occur as outcomes of a definite numbers (unlike

ii) probability that ...

that in binomial) of an experiment but which occur at random points of time and space. $\sum_{n=0}^{\infty}$ → poisson distribution.

Note:- Difference between Binomial & poisson distribution.

1. The binomial distribution is characterised by two parameters p and 'n' while the poisson distribution is characterised by a single parameter λ.

2. The sample space for the Binomial distribution is $\{0, 1, 2 \ldots n\}$ while the sample space for the poisson distribution is $\{0, 1, 2 \ldots n\}$.

2m → Mean = variance, poisson distribution.

Moments of the poisson distribution.

[Expected value - Mean.

Let x be a poisson dis random variable with parameter λ, then

$$P[X = x] = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$(M_1)\ E(X) = \bar{x} = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\left[\frac{0.\lambda^0}{0!} + \frac{1.\lambda^1}{1!} + \frac{2.\lambda^2}{2!} + \frac{3.\lambda^3}{3!} + \ldots\right]$$

$$= \lambda \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x-1}}{x(x-1)!} = e^{-\lambda}\left[\frac{\lambda}{1!} + \frac{2\lambda^2}{2!} + \frac{3\lambda^3}{3!} \ldots\right]$$

$$= e^{-\lambda}\lambda\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots\right]$$

$$= \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \qquad = e^{-\lambda} \cdot \lambda \cdot e^{\lambda}$$

$$= \lambda.$$

$$= \lambda e^{-\lambda} \left\{1 + \lambda + \frac{\lambda^2}{2!} + \ldots\right\}$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda.$$

$$E(X^2) = \sum_{x=0}^{\infty} x^2 \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

(μ_2')

$$= \sum_{x=0}^{\infty} \{x(x-1)+x\} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} + \underbrace{\sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^x}{x!}}_{\lambda}$$

$$= e^{-\lambda}\lambda^2 \sum_{x=0}^{\infty} \frac{x(x-1)\lambda^{x-2}}{x(x-1)(x-2)!} +$$

$$e^{-\lambda}\sum_{x=0}^{\infty} \frac{\lambda^x x}{x!} = e^{-\lambda}\lambda^2\left[1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\right]$$

$$= e^{-\lambda}\lambda^2\left\{\sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda\right]$$

$$= e^{-\lambda}\lambda^2 e^{\lambda} + \lambda = \lambda^2+\lambda.$$

$$\mu_3' = E(X^3) = \sum_{x=0}^{\infty} x^3 \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \{x(x-1)(x-2)+3x(x-1)+x\}\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)(x-2)\frac{e^{-\lambda}\lambda^x}{x!} +$$

$$3\sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{x(x-1)(x-2)e^{-\lambda}\lambda^x}{x(x-1)(x-2)(x-3)!} +$$

$$3\sum_{x=0}^{\infty} \frac{x(x-1)e^{-\lambda}\lambda^x}{x(x-1)(x-2)!} + E(x)$$

ii) probability that atleast 3 numbers busy is

$$= \lambda^3 \sum_{x=3}^{\infty} \frac{e^{-\lambda} \lambda^{x-3}}{(x-3)!} + 3\lambda^2 \sum_{x=2}^{\infty} \frac{e^{-\lambda} \lambda^{x-2}}{(x-2)!} + \lambda$$

$$= \lambda^3 e^{-\lambda} \sum_{x=3}^{\infty} \frac{\lambda^{x-3}}{(x-3)!} + 3\lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$= \lambda^3 e^{-\lambda}\left[1+\lambda+\frac{\lambda^2}{2!}+\cdots\right] + 3\lambda^2 + \lambda.$$

$$\mu_4' = \sum_{x=0}^{\infty} x^4 \frac{e^{-\lambda}\lambda^x}{x!} \qquad = \lambda^3 e^{-\lambda} \cdot e^{\lambda} + 3\lambda^2 + \lambda.$$

$$= \lambda^3 + 3\lambda^2 + \lambda.$$

$$= \sum_{x=0}^{\infty} \{x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x\} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \lambda^4 e^{-\lambda} \sum_{x=0}^{\infty} \frac{x(x-1)(x-2)(x-3)\lambda^{x-4}}{x(x-1)(x-2)(x-3)(x-4)!}$$

$$+ 6\lambda^3 e^{-\lambda} \sum_{x=0}^{\infty} \frac{x(x-1)(x-2)\lambda^{x-3}}{x(x-1)(x-2)(x-3)!}$$

$$+ 7\lambda^2 e^{-\lambda} \sum_{x=0}^{\infty} \frac{x(x-1)\lambda^{x-2}}{x(x-1)(x-2)!} +$$

$$\sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \lambda^4 e^{-\lambda} \sum_{x=4}^{\infty} \frac{\lambda^{x-4}}{(x-4)!} + 6\lambda^3 e^{-\lambda} \sum_{x=3}^{\infty} \frac{\lambda^{x-3}}{(x-3)!}$$

$$+ 7\lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + E(x)$$

$$= \lambda^4 . e^{-\lambda} e^{\lambda} + 6\lambda^3 . e^{-\lambda} . e^{\lambda}.$$

$$+ 7\lambda^2 . e^{-\lambda} . e^{\lambda} + e^{\lambda}.$$

$$\mu_4' = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

The four central moments are obtained as follows:

... of fifteen called between
week days is ...

Always:- $\mu_1 = 0$

$$\mu_2 = \mu_2' - (\mu_1')^2$$
$$= \lambda^2 + \lambda - (\lambda)^2$$
$$= \lambda^2 + \lambda - \lambda^2$$
$$\mu_2 = \lambda.$$

2m

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$
$$= \lambda^3 - 3\lambda^2 + \lambda - 3(\lambda^2+\lambda)\lambda + 2(\lambda)^3$$
$$= \lambda^3 + 3\lambda^2 + \lambda - 3\lambda^3 - 3\lambda^2 + 2\lambda^3.$$
$$\mu_3 = \lambda.$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$
$$- 4(\lambda^3 + 3\lambda^2 + \lambda)\lambda + 6(\lambda^2 + \lambda)\lambda^2 - 3\lambda^4.$$
$$= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda - 4\lambda^4 - 12\lambda^3 - 4\lambda$$
$$+ 6\lambda^4 + 6\lambda^3 - 3\lambda^4$$
$$= 7\lambda^4 + 12\lambda^3 + 3\lambda^2 - 7\lambda^4 - 12\lambda^3 + \lambda$$
$$= 3\lambda^2 + \lambda.$$

Skewness $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{\lambda^2}{\lambda^3} = \dfrac{1}{\lambda}.$

Kurtosis $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{3\lambda^2 + \lambda}{\lambda^2}$

$$= 3 + \frac{1}{\lambda}$$

$$\gamma_1 = \sqrt{\beta_1} = \frac{1}{\lambda}$$

$$\gamma_2 = \beta_2 - 3 = 3 + \frac{1}{\lambda} - 3 = \frac{1}{\lambda}.$$

RECURRENCE FORMULA FOR MOMENTS:-

Recurrence formula $\mu_{r+1} = \lambda r \mu_{r-1} + \lambda \frac{d\mu_r}{d\lambda}$

$$\mu_r = E[(x-\mu)^r]$$

For poisson distribution $\mu = \lambda$.

$$\mu_r = E[(x-\lambda)^r]$$

$$= \sum_{x=0}^{\infty} (x-\lambda)^r . p(x).$$

$$= \sum_{x=0}^{\infty} (x-\lambda)^r . \frac{e^{-\lambda}\lambda^x}{x!}$$

Differentiating with respect to 'λ'

$$\frac{d\mu_r}{d\lambda} = -\sum_{x=0}^{\infty} r(x-\lambda)^{r-1} . \frac{e^{-\lambda}.\lambda^x}{x!}$$

$$+ \sum_{x=0}^{\infty} \frac{(x-\lambda)^r}{x!} [e^{-\lambda} x \lambda^{x-1} - e^{-\lambda}\lambda^x]$$

$$= -r \underbrace{\sum_{x=0}^{\infty} (x-\lambda)^{r-1} \frac{e^{-\lambda}.\lambda^x}{x!}}_{}^{\mu_{r-1}}$$

$$+ \sum_{x=0}^{\infty} \frac{(x-\lambda)^r}{x!} [\lambda^{x-1} e^{-\lambda}(x-\lambda)]$$

$$= -r.\mu_{r-1} + \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{\lambda x!} (x-\lambda)^{r+1}$$

$$= -r . \mu_{r-1} + \frac{1}{\lambda} . \mu_{r+1}$$

(i.e) $\frac{d\mu_r}{d\lambda} = -r . \mu_{r-1} + \frac{1}{\lambda} \mu_{r+1}$

$$\frac{d\mu_r}{d\lambda} + r . \mu_{r-1} = \frac{1}{\lambda}\mu_{r+1}$$

$$\Rightarrow \lambda\left[\frac{d\mu_r}{d\lambda} + r . \mu_{r-1}\right] = \mu_{r+1}.$$

(i.e) $\cdot \mu_{r+1} = \lambda r . \mu_{r-1} + \lambda . \frac{d\mu_r}{d\lambda}$

From this formula we can find the first four central moments

put $r=1$,

$$\mu_2 = \lambda . \mu_0 + \lambda \frac{d\mu_1}{d\lambda}$$

$\mu_0 = 1$ & $\mu_1 = 0$

$\therefore \mu_2 = \lambda.$

put $r=2$,

$$\mu_3 = 2\lambda . \mu_1 + \lambda . \frac{d\mu_2}{d\lambda}$$

$$= 0 + \lambda\left(\frac{\cancel{d\lambda}}{\cancel{d\lambda}}\right) = \lambda.$$

$\therefore \mu_3 = \lambda.$

put $r=3$,

$$\mu_4 = 3\lambda . \mu_2 + \lambda . \frac{d\mu_3}{d\lambda}$$

$$= 3.\lambda.\lambda + \lambda\left(\frac{\cancel{d\lambda}}{\cancel{d\lambda}}\right)$$

$$\mu_4 = 3\lambda^2 + \lambda.$$

MOMENTS GENERATING FUNCTION OF THE POISSON DISTRIBUTION

Let x be a random variable following poisson distribution with parameter λ, then the M.G.F is

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx} . p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} . \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} e^{-\lambda} \frac{(\lambda . e^t)^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda . e^t)^x}{x!}$$

$$= e^{-\lambda} \left\{ 1 + \lambda . e^t + \frac{(\lambda . e^t)^2}{2!} + \frac{(\lambda . e^t)^3}{3!} - \lambda \ldots \right\}$$

$$= e^{\lambda(e^t - 1)} \quad = e^{-\lambda}\left(e^{\lambda . e^t}\right) \quad e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$= e^{\lambda(e^t - 1)}$$

$$M_x(t) = e^{\lambda(e^t - 1)}$$

$$\mu_r' = \left[\frac{d^r}{dt^r} M_x(t)\right]_{t=0}$$

when $r = 1$,

$$\mu_1' = \left[\frac{d}{dt} e^{\lambda(e^t - 1)}\right]_{t=0}.$$

$$= \left[\lambda . e^t . e^{\lambda(e^t - 1)}\right]_{t=0}$$

$$= \lambda.$$

when $r = 2$,

$$\mu_2' = \left[\frac{d^2}{dt^2} e^{\lambda(e^t - 1)}\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left\{\frac{d}{dt} e^{\lambda(e^t - 1)}\right\}\right]_{t=0}$$

$$= \lambda\left[\frac{d}{dt}\left\{e^t e^{\lambda(e^t - 1)}\right\}\right]_{t=0}$$

$$= \lambda\left[e^t . e^{\lambda(e^t - 1)} + \lambda e^t . e^t . e^{\lambda(e^t - 1)}\right]_{t=0}$$

$$= \lambda[1 + \lambda] = \lambda + \lambda^2$$

$$\mu_3' = \left[\frac{d^3}{dt^3} M_x(t)\right]$$

$$= \left[\frac{d}{dt}\left\{\frac{d^2}{dt^2} M_X(t)\right\}\right]_{t=0}$$

$$= \lambda \frac{d}{dt}\left[e^t . e^{\lambda(e^t-1)} + \lambda . e^{2t} . e^{\lambda(e^t-1)}\right]_{t=0}$$

$$= \lambda . \Big[e^t e^{\lambda(e^t-1)} + \lambda e^{2t} e^{\lambda(e^t-1)} + 2\lambda . e^{2t} e^{\lambda(e^t-1)} + \lambda e^{2t} . \lambda e^t e^{\lambda(e^t-1)}\Big]_{t=0}$$

$$= \lambda[1 + \lambda + 2\lambda + \lambda^2]$$

$$= \lambda . [1 + 3\lambda + \lambda^2]$$

$$= \lambda + 3\lambda^2 + \lambda^3 .$$

$$\mu_4' = \left[\frac{d}{dt}\left\{\frac{d^3}{dt^3} M_X(t)\right\}\right]_{t=0}$$

$$= \lambda\left[\frac{d}{dt}\Big\{e^t . e^{\lambda(e^t-1)} + \lambda e^{2t} e^{\lambda(e^t-1)} + 2\lambda . e^{2t} e^{\lambda(e^t-1)} + \lambda^2 e^{3t} e^{\lambda(e^t-1)}\Big\}\right]_{t=0}$$

$$= \lambda\Big[e^t e^{\lambda(e^t-1)} + \lambda . e^{2t} e^{\lambda(e^t-1)} + 2\lambda e^{2t} e^{\lambda(e^t-1)} + \lambda^2 e^{3t} e^{\lambda(e^t-1)} + 4\lambda e^{2t} e^{\lambda(e^t-1)} + 2\lambda e^{2t} \lambda e^t e^{\lambda(e^t-1)} + 3\lambda^2 e^{3t} e^{\lambda(e^t-1)} + \lambda^2 e^{3t} \lambda e^t e^{\lambda(e^t-1)}\Big]$$

$$= \lambda[1 + \lambda + 2\lambda + \lambda^2 + 4\lambda + 2\lambda^2 + 3\lambda^2 + \lambda^3]$$

$$= \lambda[1 + 7\lambda + 6\lambda^2 + \lambda^3]$$

$$= \lambda + 7\lambda^2 + 6\lambda^3 + \lambda^4 .$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$\mu_3 = \mu_3' - 3\mu_2' \mu_1' + 2(\mu_1')^3$$

$$= \lambda^3 + 3\lambda^2 + \lambda - 3(\lambda^2 + \lambda)\lambda + 2\lambda^3 .$$

$= \lambda^3 + 3\lambda^2 + \lambda - 3\lambda - 3\lambda^2 + 2\lambda^3.$

$\mu_3 = \lambda$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda - 4(\lambda^3 + 3\lambda^2 + \lambda)\lambda + 6(\lambda^2 + \lambda)\lambda^2 - 3\lambda^4$$

$$= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda - 4\lambda^4 - 12\lambda^3 - 4\lambda^2 + 6\lambda^4 + 6\lambda^3 - 3\lambda^4$$

$$= 3\lambda^2 + \lambda$$

$$\mu_4 = 3\lambda^2 + \lambda$$

Additive property (or) Reproductive property of Independent poisson variates

If X_1 & X_2 are two independent poisson variates with parameters λ_1 & λ_2 respectively then their sum $(X_1 + X_2)$ is also a poisson variate with parameters $(\lambda_1 + \lambda_2)$.

If X is a poisson variate with parameter λ then

$$P(X = x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

and its MGF is

$$M_X(t) = e^{\lambda(e^t - 1)}$$

$$M_{X_1}(t) = e^{\lambda_1(e^t - 1)}.$$

$$M_{X_2}(t) = e^{\lambda_2(e^t - 1)}$$

MGF of the sum is

$$M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t)$$

$$= e^{\lambda_1(e^t - 1)} \cdot e^{\lambda_2(e^t - 1)}$$

$= e^{\lambda_1+\lambda_2 (e^t-1)}$

which is the M.G.F of a random variate following poisson distribution with parameter $\lambda_1+\lambda_2$. Hence by the uniqueness theorem of M.G.F, X_1+X_2 is also a poisson variate with parameter $\lambda_1+\lambda_2$.

This result can also be g.

for X_i, $i=1,2\ldots n$ independent poisson variates when parameters λ_i, $i=1,2\ldots n$

$$M_{X_i}(t) = e^{\lambda_i(e^t-1)}$$

$$M_{X_1+X_2+\ldots+X_n} = M_{X_1}(t)\cdot M_{X_2}(t)\ldots M_{X_n}(t)$$

$$M_{X_1+X_2+\ldots+X_n}(t) = e^{\lambda_1(e^t-1)}\cdot e^{\lambda_2(e^t-1)}\ldots e^{\lambda_n(e^t-1)}$$

$$= e^{\lambda_1+\lambda_2+\ldots+\lambda_n(e^t-1)}$$

which is the M.G.F of a poisson variate with parameter $\lambda_1+\lambda_2+\ldots+\lambda_n$

Hence by uniqueness theorem of M.G.F is $\sum_{i=1}^{n} X_i$ is also a poisson variate with parameter $\sum_{i=1}^{n}\lambda_i$.

The difference of two independent poisson variate is not a poisson variate

$$M_{X_1-X_2}(t) = M_{(X_1+(-X_2))}(t) = M_{X_1}(t)\cdot M_{X_2}(-t)$$

$$= e^{\lambda_1(e^t-1)}\cdot e^{\lambda_2(e^{-t}-1)}$$

$$= e^{\lambda_1(e^t-1)+\lambda_2(e^{-t}-1)}$$

which cannot be put in the form $e^{\lambda(e^t-1)}$.

Hence (X_1-X_2) is not a poisson variate.

Poisson variate is always non-negative.

MODE OF THE POISSON DISTRIBUTION

Let $P(x)$ and $P(x-1)$ be the probabilities when the variate takes the values x & $(x-1)$ respectively

$$P(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$P(x-1) = \frac{e^{-\lambda} \cdot \lambda^{x-1}}{(x-1)!}$$

$$\frac{P(x)}{P(x-1)} = \frac{e^{-\lambda} \cdot \lambda^x}{x!} \times \frac{(x-1)!}{e^{-\lambda} \cdot \lambda^{x-1}}$$

$$= \frac{\cancel{e^{-\lambda}} \cdot \lambda^x}{x\cancel{(x-1)!}} \cdot \frac{\cancel{(x-1)!}}{\cancel{e^{-\lambda}} \cdot \lambda^{x-1}}$$

$$\frac{P(x)}{P(x-1)} = \frac{\lambda}{x}$$

We discuss the following cases:-

Case 1:- When λ is not an integer. Let us suppose that S is the integral part of λ.

$$\frac{P(1)}{P(0)} > 1, \frac{P(2)}{P(1)} > 1, \ldots \frac{P(S-1)}{P(S-2)} > 1, \frac{P(S)}{P(S-1)} > 1$$

$$\& \frac{P(S+1)}{P(S)} < 1, \frac{P(S+2)}{P(S+1)} \ldots$$

i.e, $P(1) > P(0)$, $P(2) > P(1)$, ... $P(S-1) > P(S-2)$

$P(S) > P(S-1)$; $P(S+1) < P(S)$; $P(S+2)$...

i.e, $P(0) < P(1)$; $P(1) < P(2)$; ...

$P(S-2) < P(S-1);\ P(S-1) < P(S);\ \ldots$

$P(S) > P(S+1);\ P(S+1) > P(S+2);\ \ldots$

This shows that $P(S)$ is the maximum value. Hence in this case the integral part of λ is the modal value.

Case II :- When λ is an integer.

Here we have

$$\frac{P(1)}{P(0)} > 1;\ \frac{P(2)}{P(1)} > 1, \ldots \frac{P(\lambda-1)}{P(\lambda-2)} > 1; \ldots$$

$$\frac{P(\lambda)}{P(\lambda-1)} = 1\ \frac{P(\lambda+1)}{P(\lambda)} < 1,\ \frac{P(\lambda+2)}{P(\lambda+1)} < 1, \ldots$$

$$\therefore P(0) < P(1) < P(2) < \cdots < P(\lambda-2) < P(\lambda-1)$$
$$= P(\lambda) > P(\lambda+1) > P(\lambda+2) > P(\lambda+3) > \ldots$$

values $P(\lambda-1)$ & $P(\lambda)$. Hence when λ is an integer, the distribution is bimodal & two modes are $\lambda-1$ and λ.

Characteristic function of P.D

$$P(X=x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

Characteristic function,

$$= E(e^{itx}) = \sum_{x=0}^{\infty} e^{itx}\, p(x)$$

$$= \sum_{x=0}^{\infty} e^{itx} \cdot \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda \cdot e^{it})^x}{x!}$$

$$= e^{-\lambda}\left[1+\lambda e^{it} + \frac{(\lambda e^{it})^2}{2!} + \frac{(\lambda e^{it})^3}{3!} + \dots\right].$$

Problems:-

1. If X is a positive random variable such that $P(X=1)=0.3$ & $P(X=2)=0.2$. Find $P(X=0)$.

If X is a poisson variate with parameter λ.

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$P(X=1);\ e^{-\lambda}\lambda = 0.3 \to (1)$$

$$P(X=2);\ \frac{e^{-\lambda}\cdot\lambda^2}{2} = 0.2 \to (2)$$

Dividing (2) by (1)

$$\frac{e^{-\lambda}\lambda^2}{2e^{-\lambda}\cdot\lambda} = \frac{0.2}{0.3} \ \times 10$$

$$\frac{\lambda}{2} = \frac{2}{3}. \qquad \lambda = \frac{4}{3}.$$

$$P(X=0) = \frac{e^{-4/3}(4/3)^0}{0!} = e^{-\lambda} = e^{-4/3}.$$

i.e $e^{-4/3} = 0.2636$

$\therefore P(X=0) = 0.2636.$

2. Find the probability that atmost 5 defective fuses will be found in a box of 200 fuses if experience shows that 2 percent of such fuses are defective, given that $e^{-4}=0.0183$.

p = probability that a fuse is defective

$$p = 2\% = \frac{2}{100};\ n = 200$$

$$\lambda = np = 200 \times \frac{2}{100} = 4.$$

$$e^{-\lambda} = 0.0183.$$

phone

Probability that atmost 5 defective fuses will be found in a box of 200 fuses $= P(X \le 5)$

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$P(X \le 5) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$
$+ P(X=5)$

$$= e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2 . e^{-\lambda}}{2!} + \frac{\lambda^3 . e^{-\lambda}}{3!} + \frac{\lambda^4 . e^{-\lambda}}{4!} + \frac{\lambda^5 . e^{-\lambda}}{5!}$$

$$= e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \frac{\lambda^5}{5!}\right]$$

$$= e^{-\lambda}\left[1 + 4 + \frac{4^2}{2} + \frac{4^3}{6} + \frac{4^4}{24} + \frac{4^5}{120}\right]$$

$$= e^{-4}\left[1 + 4 + 8 + \frac{64}{6} + \frac{256}{24} + \frac{1024}{120}\right]$$

$$= e^{-4}[13 + 10.6667 + 10.6667 + 8.5333]$$

$$= 0.0183[42.8667]$$

$$= 0.7844\,\cancel{599}\ 6061$$

$$= 0.7845$$

$P(X \le 5) = 0.7845.$

3. A poisson Distribution has a double mode of $x=1$ & $x=2$. What is the probability that x will have one or the other of these two values.

If a poisson distribution with parameter λ is bimodal then the modes are given by,

$$x = \lambda - 1 \ \& \ x = \lambda.$$

Here $\lambda - 1 = 1 \Rightarrow \lambda = 2$

Put $(X = x) = \dfrac{e^{-\lambda} \cdot \lambda^x}{x!}$

$$P(X = 1) = e^{-\lambda} \cdot \lambda = e^{-2} \cdot 2$$

$$P(X = 2) = \frac{e^{-\lambda} \cdot \lambda^2}{2!} = \frac{e^{-\lambda} \cdot 2^2}{2} = e^{-2} \cdot 2$$

$$e^{-2} = 0.135.$$

Probability that X will have one or the other of these two values.

$$= P(X=1) + P(X=2) = \frac{e^{-\lambda} \cdot \lambda}{1!} + \frac{e^{-\lambda} \cdot \lambda^2}{2!}$$

$$= e^{-2} \cdot 2 + e^{-2} \cdot 2^2 = e^{-2} \cdot 2 + e^{-2} \cdot 2$$

$$= 2[e^{-2} + e^{-2}] = 2[2 \cdot e^{-2}].$$

4. Suppose that the